Math 32A, Lecture 1
Multivariable Calculus

Sample Midterm 1

Instructions: You have 50 minutes to complete the exam. There are five problems, worth a total of fifty points. You may not use any books, notes, or calculators. Show all your work; partial credit will be given for progress toward correct solutions, but unsupported correct answers will not receive credit. Remember to make your drawings large and clear, and to label your axes.

Write your solutions in the space below the questions. If you need more space, use the back of the page. Do not turn in your scratch paper.

Name: Solutions

UID: ____________________

Section: ____________________

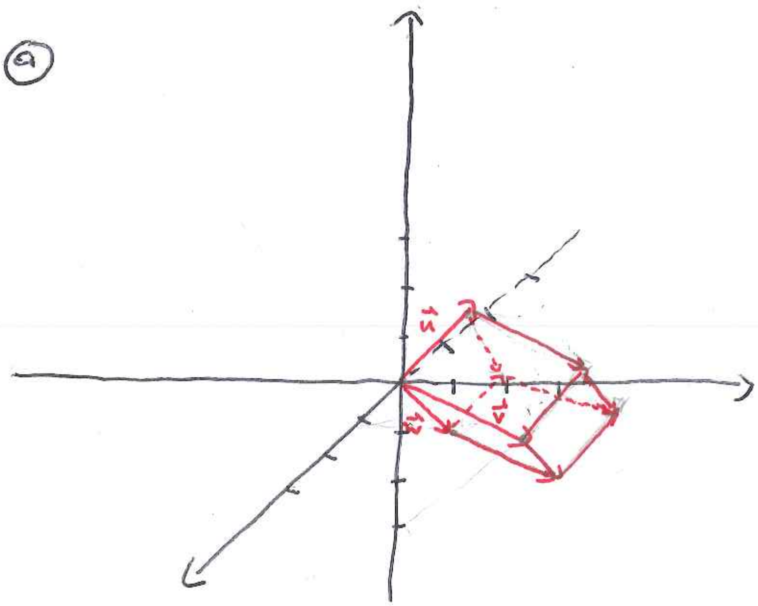
Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

Problem 1.

(a) [5pts.] Draw the parallelopiped spanned by $\mathbf{v} = \langle -3, 0, -3 \rangle$ and $\mathbf{w} = \langle 0, 1, -1 \rangle$, and $\mathbf{u} = \langle 1, 2, 2 \rangle$ in three-dimensional coordinates.

(b) [5pts.] Compute the ~~area~~ volume of this parallelopiped **using only the geometric properties of, and not the algebraic formula for** the cross product. (You can take any dot products you feel appropriate.)

(a)



(b) We want the volume $= |\vec{u} \cdot (\vec{v} \times \vec{w})|$. Now by inspection, $\langle -1, 1, 1 \rangle$ is orthogonal to both $\vec{v}$ and $\vec{w}$, so $\vec{v} \times \vec{w}$ is a scalar multiple of $\langle -1, 1, 1 \rangle$. Now if θ is the angle between $\vec{v}$ and $\vec{w}$, $\|\vec{v} \times \vec{w}\| = \|\vec{v}\| \, \|\vec{w}\| \sin\theta$. But

$$\cos\theta = \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \, \|\vec{w}\|} = \frac{3}{\sqrt{18}\sqrt{2}} = \frac{3}{\sqrt{36}} = \frac{3}{6} = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3} \Rightarrow \sin\theta = \frac{\sqrt{3}}{2}.$$

So $\|\vec{v} \times \vec{w}\| = \|\vec{v}\| \, \|\vec{w}\| \left(\frac{\sqrt{3}}{2}\right) = 6\left(\frac{\sqrt{3}}{2}\right) = 3\sqrt{3}$. So $\vec{v} \times \vec{w} = \pm \underbrace{3 \langle -1, 1, 1 \rangle}_{\text{This vector has length } 3\sqrt{3}}$

and $|\vec{u} \cdot (\vec{v} \times \vec{w})| = |\langle 1, 2, 2 \rangle \cdot (\pm 3 \langle -1, 1, 1 \rangle)| = 3\,|-1 + 2 + 2| = 9.$

Note I think this problem is slightly harder than the roughly corresponding

Problem 2.

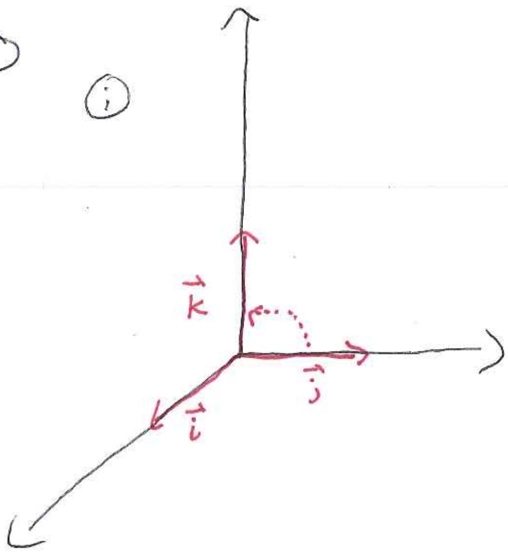
(a) [5pts.] Which of the following are right-handed systems? Justify your answers.

1. $\{\mathbf{j}, \mathbf{k}, \mathbf{i}\}$
2. $\{\mathbf{k}, -\mathbf{i}, \mathbf{j}\}$
3. $\{\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\rangle, \langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\rangle, \mathbf{k}\}$

(b) [5pts.] Demonstrate that the cross product is not associative; that is, give three specific vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ with the property that $\mathbf{v} \times (\mathbf{u} \times \mathbf{w}) \neq (\mathbf{v} \times \mathbf{u}) \times \mathbf{w}$. [Hint: You don't need to use very complicated vectors.]

(a)

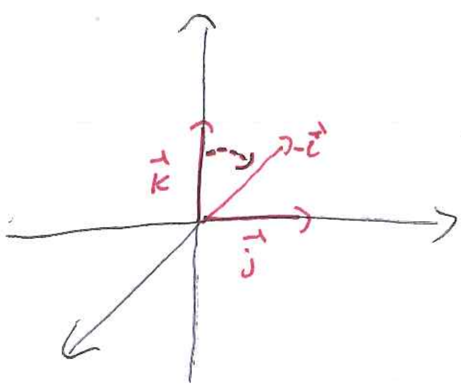
(i)



Notice that turning from $\vec{j}$ to $\vec{k}$ is counterclockwise as shown, and therefore the third vector in the system points toward the viewer. So this is a right-handed system

Alternately
$$\left.\begin{aligned} \vec{j} \times \vec{k} &= \vec{i} \\ \vec{k} \times \vec{i} &= \vec{j} \\ \vec{i} \times \vec{j} &= \vec{k} \end{aligned}\right\} \checkmark$$

(ii)

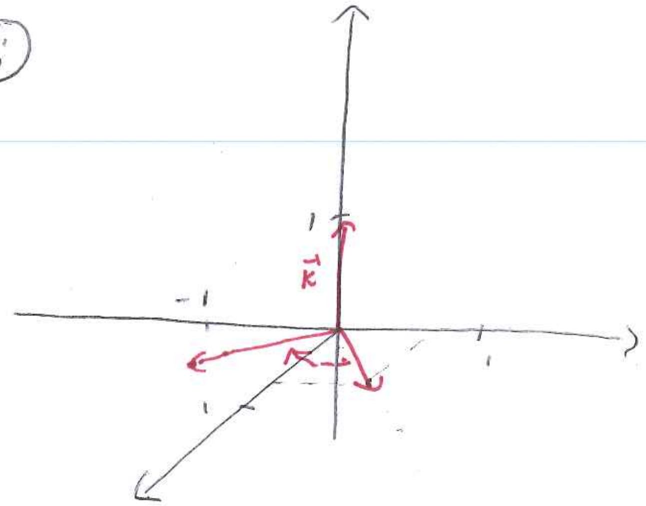


Viewed downward from $\vec{j}$, turning from $\vec{k}$ to $-\vec{i}$ is clockwise, so this is not a right-handed system.

Alternately $\vec{k} \times (-\vec{i}) = -\vec{k} \times \vec{i}$
$$= -\vec{j}$$

showing this is not right-handed.

(iii)



Note $\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \rangle$ and $\langle \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0 \rangle$ are unit vectors in the xy-plane. Viewed downward from $\vec{k}$, turning from $\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \rangle$ to $\langle \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0 \rangle$ is clockwise, so this is <u>not</u> a right-handed system

(b) Consider $\vec{u} = \vec{i}$, $\vec{v} = \vec{i}$, $\vec{w} = \vec{j}$

$$\vec{v} \times (\vec{u} \times \vec{w}) = \vec{i} \times (\vec{i} \times \vec{j}) = \vec{i} \times \vec{k} = -\vec{j}$$

$$(\vec{v} \times \vec{u}) \times \vec{w} = (\vec{i} \times \vec{i}) \times \vec{j} = (\vec{0}) \times \vec{j} = \vec{0}$$

[Remember the cross product of two copies of the same vector, or more generally two vectors which are scalar multiples, is zero.]

Problem 3.

(a) [5pts.] Give an equation for the plane P that contains the lines $\mathbf{r}_1(t) = \langle 2,1,0\rangle + t\langle 1,2,3\rangle$ and $\mathbf{r}_2(s) = \langle 5,2,8\rangle + s\langle 3,1,8\rangle$.

(b) [5pts.] For each of the two lines below, decide whether the line is contained in P, intersects P in a single point, or does not intersect P. If you decide that the second option is correct, give the point of intersection.

$$\mathbf{r}_3(u) = \langle 1,2,0\rangle + u\langle 1,-3,2\rangle$$
$$\mathbf{r}_4(v) = \langle 6,5,-8\rangle + v\langle 2,2,-4\rangle$$

(a) The plane contains the direction vectors $\vec{v}_1 = \langle 1,2,3\rangle$ and $\vec{v}_2 = \langle 3,1,8\rangle$, so a normal vector is given by

$$\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 3 & 1 & 8 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 1 & 8 \end{vmatrix}\vec{i} - \begin{vmatrix} 1 & 3 \\ 3 & 8 \end{vmatrix}\vec{j} + \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}\vec{k}$$

$$= (16-3)\vec{i} - (8-9)\vec{j} + (1-6)\vec{k}$$

$$= 13\vec{i} + \vec{j} - 5\vec{k}$$

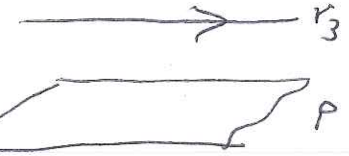
So an equation for P looks like $13x + y - 5z = d$.

Since P contains $(2,1,0)$, $d = 13(2) + 1 - 5(0) = 27$.

$$\boxed{13x + y - 5z = 27}$$

$\vec{r}_3$) The direction vector $\langle 1, -3, 2\rangle$ is orthogonal to $\vec{n} = \langle 13, 1, -5\rangle$ (since $13(1) + 1(-3) + -5(2) = 13 - 3 - 10 = 0$), so $\vec{r}_3$ is at least parallel to some line in P.

However $(1, 2, 0)$ is on $\vec{r}_3$ but not in P, since $13(1) + 2 - 5(0) = 15$. So $\vec{r}_3$ parametrizes a line that does not intersect P.



$\vec{r}_4$) Parametric equations for $\vec{r}_4$ are

$$x(v) = 6 + 2v$$
$$y(v) = 5 + 2v$$
$$z(v) = -8 - 4v$$

Is there a value of v such that these lie on P?

$$13(6 + 2v) + (5 + 2v) - 5(-8 - 4v) = 27$$

$$78 + 26v + 5 + 2v + 40 + 20v = 27$$

$$48v = -96$$

$$\boxed{v = -2}$$

So the line and the plane intersect at $v = -2 \rightsquigarrow$ at $\langle 2, 1, 0\rangle$.

Problem 4.

(a) [5pts.] Sketch the plane curves $\mathbf{c}(t) = \langle t^3 + 4t^2, t^2 - 1\rangle$ and $\mathbf{d}(s) = \langle s^2 - 1, s\rangle$ and find their points of intersection.(other than $\langle 0, -1\rangle$).

(b) [5pts.] The *angle between two curves* is the angle between their tangent lines at the point of intersection. With this in mind, decide what the cosine of the angle between the two curves in part (a) is at each of the intersection points you found above.

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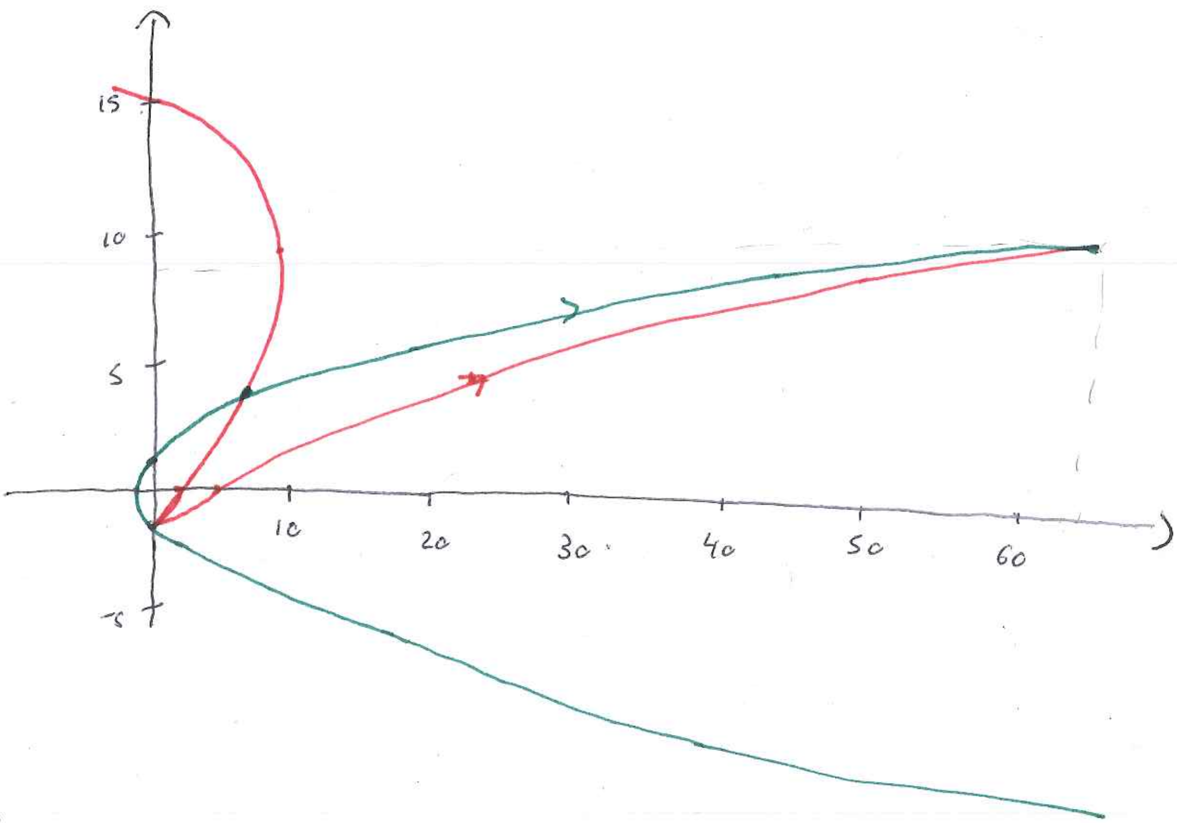


$$\begin{cases} t^3 + 4t^2 = s^2 - 1 \\ t^2 - 1 = s \end{cases} \rightsquigarrow \text{substitute}$$

$$t^3 + 4t^2 = (t^2-1)^2 - 1$$

$$t^3 + 4t^2 = t^4 - 2t^2 + 1 - 1$$

$$0 = t^4 - 6t^2 - t^3$$

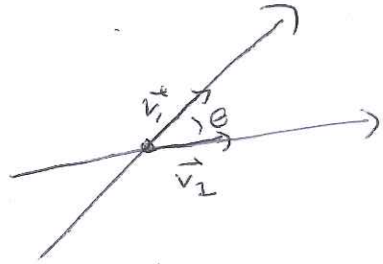
$$= t^2(t^2 - t - 6)$$

$$= t^2(t-3)(t+2)$$

$t = 0 \quad t = 3 \quad t = -2$

Intersection Points	$\langle 0, -1\rangle$	$\langle 63, 8\rangle$	$\langle 8, 3\rangle$
	$t = 0$, $s = -1$	$t = 3$, $s = 8$	$t = -2$, $s = 3$

(b) The angle between two lines is the intersection between their direction vectors.



The direction vector of the tangent line to a curve $\vec{c}(t)$ at t_0 is $\vec{c}\,'(t_0)$.

So $\vec{c}\,'(t) = \langle 3t^2+8t, 2t\rangle$

$\vec{d}\,(s) = \langle 2s, 1\rangle$

At $\langle 63, 8\rangle$

$$\vec{v}_1 = \vec{c}\,'(3) = \langle 27+24, 6\rangle = \langle 51, 6\rangle$$

$$\vec{v}_2 = \vec{d}\,'(8) = \langle 16, 1\rangle$$

$$\cos\theta = \frac{\vec{v}_1 \cdot \vec{v}_2}{\|\vec{v}_1\|\,\|\vec{v}_2\|} = \frac{822}{\sqrt{2637}\ \sqrt{257}}$$

At $\langle 8, 3\rangle$

$$\vec{v}_1 = \vec{c}\,'(-2) = \langle 12-16, -4\rangle = \langle -4, -4\rangle$$

$$\vec{v}_2 = \vec{d}\,'(3) = \langle 6, 1\rangle$$

$$\cos\theta = \frac{-24-4}{\sqrt{32}\ \sqrt{37}} = \frac{-28}{\sqrt{32}\sqrt{37}}$$

Problem 5.

Consider the vector-valued function $\mathbf{r}(t) = \langle \cos(t), \cos(2t), \sin(t) \rangle$.

(a) [5pts.] Draw the projections of $\mathbf{r}(t)$ to the three coordinate planes, and use these to give a sketch of the space curve determined by $\mathbf{r}(t)$.

(b) [5pts.] Find the equation of the tangent line to $\mathbf{r}(t)$ at $t_0 = \pi$.

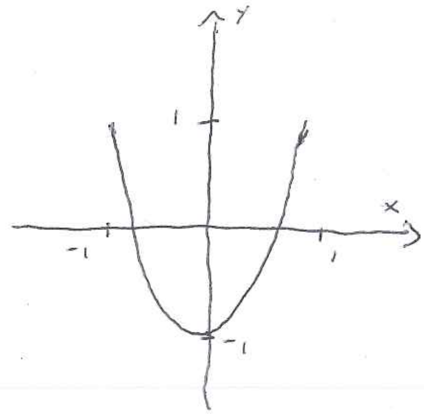
(a) xy-plane

$\langle \cos t, \cos(2t) \rangle$

$= \langle \cos t, 2\cos^2 t - 1 \rangle$

↑

parabola!

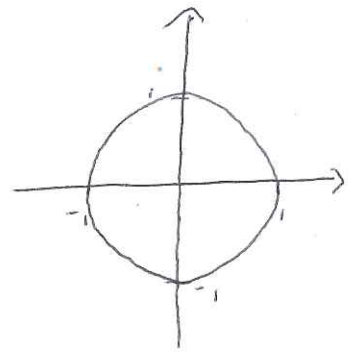


xz-plane

$\langle \cos t, \sin t \rangle$

↑

circle



yz-plane

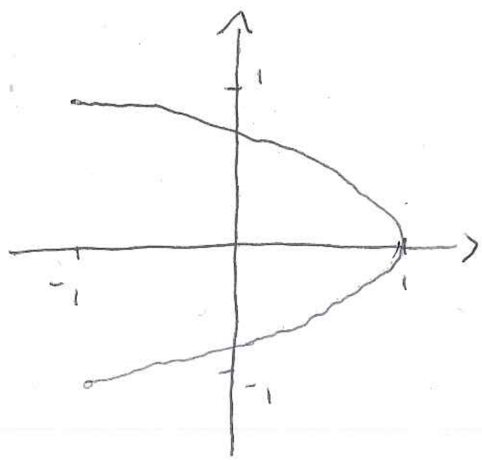
$\langle \cos(2t), \sin(t) \rangle$

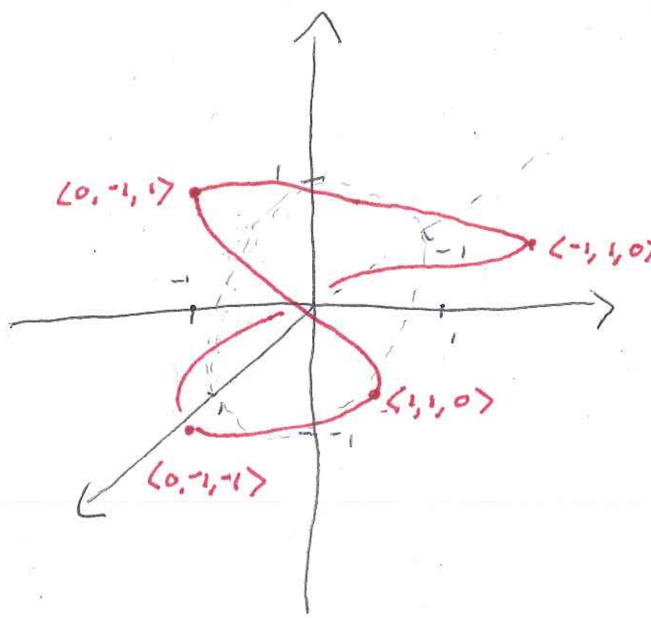
$\shortparallel$

$\langle 1 - 2\sin^2 t, \sin t \rangle$

↑

parabola!





(b) $\vec{r}(t_0) = \langle \cos(\pi), \cos(2\pi), \sin(\pi) \rangle = \langle -1, 1, 0 \rangle$

$\vec{r}\,'(t) = \langle -\sin(t), -2\sin(2t), \cos t \rangle$

$\vec{r}\,'(t_0) = \langle 0, 0, -1 \rangle$

Tangent Line $\vec{\ell}(s) = \langle -1, 1, 0 \rangle + s\langle 0, 0, -1 \rangle$